disparity might be due, in part, to the range of stress involved. He states—

* * * during the pumping test of 1940 the head in the Lloyd sand declined to a new low over a considerable area in the vicinity of the pumped wells, and consequently the stress in the skeleton of the aquifer reached a new high. It is to be expected that the modulus of elasticity would be smaller for the new, higher range of stress than for the old range over which the stress had fluctuated many times.

**AQUIFER TESTS—BASIC THEORY**

**WELL METHODS—POINT SINK OR POINT SOURCE**

**CONSTANT DISCHARGE OR RECHARGE WITHOUT VERTICAL LEAKAGE**

**EQUILIBRIUM FORMULA**

Wenzel (1942, p. 79–82) showed that the equilibrium formulas used by Slichter (1899), Turneaure and Russell (1901), Israelson (1950), and Wyckoff, Botset, and Muskat (1932) are essentially modified forms of a method developed by Thiem (1906), as are the formulas developed by Dupuit (1848) and Forchheimer (1901). Thiem apparently was the first to use the equilibrium formula for determining permeability and it is frequently associated with his name. The formula was developed by Thiem from Darcy's law and provides a means for determining aquifer transmissibility if the rate of discharge of a pumped well and the drawdown in each of two observation wells at different known distances from the pumped well are known. The Thiem formula, in nondimensional form, can be written as

\[
T = \frac{Q \log_e \left( \frac{r_2}{r_1} \right)}{2\pi(s_1 - s_2)},
\]

where the subscript \( e \) in the log term indicates the natural logarithm. In the usual Geological Survey units (see p. 73), and using common logarithms, equation 1 becomes

\[
T = \frac{527.7 Q \log_{10} \left( \frac{r_2}{r_1} \right)}{s_1 - s_2},
\]

where

- \( T \) = coefficient of transmissibility, in gallons per day per foot,
- \( Q \) = rate of discharge of the pumped well, in gallons per minute,
- \( r_1 \) and \( r_2 \) = distances from the pumped well to the first and second observation wells, in feet, and
- \( s_1 \) and \( s_2 \) = drawdowns in the first and second observation wells, in feet.

The derivation of the formula is based on the following assumptions:

(a) the aquifer is homogeneous, isotropic, and of infinite areal extent;
(b) the discharging well penetrates and receives water from the entire thickness of the aquifer;
(c) the coefficient of transmissibility is constant at all times and at all places;
(d) pumping has continued at a
uniform rate for sufficient time for the hydraulic system to reach a steady-state (i.e., no change in rate of drawdown as a function of time) condition; and (e) the flow is laminar. The formula has wide application to ground-water problems despite the restrictive assumptions on which it is based.

The procedure for application of equation 2 is to select some convenient elapsed pumping time, \( t \), after reaching the steady-state condition, and on semilog coordinate paper plot for each observation well the drawdowns, \( s \), versus the distances, \( r \). By plotting the values of \( s \) on the arithmetic scale and the values of \( r \) on the logarithmic scale, the observed data should lie on a straight line for the equilibrium formula to apply. From this straight line an arbitrary choice of \( s_1 \) and \( s_2 \) should be made and the corresponding values of \( r_1 \) and \( r_2 \) recorded. Equation 2 can then be solved for \( T \).

Jacob (1950, p. 368) recognized that the coefficient of storage could also be determined if the hydraulic system had reached a steady-state condition (see assumption d, above), for thereafter the drawdown is expressed very closely by the nondimensional formula

\[
s = \frac{Q}{4\pi T} \log_e \frac{2.25Tt}{r^2S} \tag{3}
\]

or, in the usual Survey units and using common logarithms,

\[
s = \frac{264Q}{T} \log_{10} \frac{0.3Tt}{r^2S} \tag{4}
\]

Thus after the coefficient of transmissibility has been determined, the coordinates of any point on the semilogarithmic graph previously described can be used to solve equation 4 for the coefficient of storage.

NONEQUILIBRIUM FORMULA

Theis (1935) derived the nonequilibrium formula from the analogy between the hydrologic conditions in an aquifer and the thermal conditions in an equivalent thermal system. The analogy between the flow of ground water and heat conduction for the steady-state condition has been recognized at least since the work of Slichter (1899), but Theis was the first to introduce the concept of time to the mathematics of ground-water hydraulics. Jacob (1940) verified the derivation of the nonequilibrium formula directly from hydraulic concepts.

The nonequilibrium formula in nondimensional form is

\[
s = \frac{Q}{4\pi T} \int_{r_S/4Tt}^{\infty} \frac{e^{-u}}{u} du, \tag{5}
\]

where \( u = r^2S/4Tt \), and where the integral expression is known as an exponential integral.
Using the ordinary Survey units equation 5 may be written as

\[ s = \frac{114.6Q}{T} \int_{1.87r^2S/Tt}^{\infty} \frac{e^{-u}}{u} \, du, \]  

(6)

where
\[ u = \frac{1.87r^2S}{Tt}, \]
\[ s = \text{drawdown, in feet, at any point of observation in the vicinity of a well discharging at a constant rate}, \]
\[ Q = \text{discharge of a well, in gallons per minute}, \]
\[ T = \text{transmissibility, in gallons per day per foot}, \]
\[ r = \text{distance, in feet, from the discharging well to the point of observation}, \]
\[ S = \text{coefficient of storage, expressed as a decimal fraction}, \]
\[ t = \text{time in days since pumping started}. \]

The nonequilibrium formula is based on the following assumptions:
(a) the aquifer is homogeneous and isotropic; (b) the aquifer has infinite areal extent; (c) the discharge or recharge well penetrates and receives water from the entire thickness of the aquifer; (d) the coefficient of transmissibility is constant at all times and at all places; (e) the well has an infinitesimal (reasonably small) diameter; and (f) water removed from storage is discharged instantaneously with decline in head. Despite the restrictive assumptions on which it is based, the nonequilibrium formula has been applied successfully to many problems of ground-water flow.

The integral expression in equation 6 cannot be integrated directly, but its value is given by the series

\[ \int_{1.87r^2S/Tt}^{\infty} \frac{e^{-u}}{u} \, du = W(u) = -0.577216 - \log u + u \]
\[ - \frac{u^2}{2.2!} + \frac{u^3}{3.3!} - \frac{u^4}{4.4!} + \ldots \]  

(7)

where, as already indicated,
\[ u = \frac{1.87r^2S}{Tt} \]  

(8)

The exponential integral is written symbolically as \( W(u) \) which is read "well function of \( u \)." Values of \( W(u) \) for values of \( u \) from \( 10^{-15} \) to 9.9, as tabulated in Wenzel (1942), are given in table 2. In order to determine the value of \( W(u) \) for a given value of \( u \), using table 2, it is necessary to express \( u \) as some number \( (N) \) between 1.0 and 9.9, multiplied by 10 with the appropriate exponent. For example, when \( u \) has a value of 0.0005 (that is, \( 5.0 \times 10^{-4} \)), \( W(u) \) is
determined from the line $N=5.0$ and the column $N \times 10^{-4}$ to be 7.0242.

Referring to equations 6 and 8, if $s$ can be measured for one value of $r$ and several values of $t$, or for one value of $t$ and several values of $r$, and if the discharge $Q$ is known, then $S$ and $T$ can be determined. Once these aquifer constants have been determined, it is possible, theoretically, to compute the drawdown for any time at any point on the cone of depression for any given rate and distribution of pumping from wells. It is not possible, however, to determine $T$ and $S$ directly from equation 6, because $T$ occurs in the argument of the function and again as a divisor of the exponential integral. Thes devised a convenient graphical method of superposition that makes it possible to obtain a simple solution of the equation.

The first step in this method is the plotting of a type curve on logarithmic coordinate paper. From table 2 values of $W(u)$ have been plotted against the argument $u$ to form the type curve shown in figure 23. It is shown in two segments, A–A and B–B, in order that the portion of the type curve necessary in the analysis of pumping test data could be plotted on a sheet of convenient size. Curve B–B is an extension of curve A–A and overlaps curve A–A for values of $W(u)$ from about 0.22 to 1.0.

Rearranging equations 6 and 8 there follows

$$s = \left[ \frac{114.6Q}{T} \right] W(u)$$

or

$$\log s = \left[ \log \frac{114.6Q}{T} \right] + \log W(u)$$

and

$$\frac{r^2}{t} = \left[ \frac{T}{1.87S} \right] u$$

or

$$\log \frac{r^2}{t} = \left[ \log \frac{T}{1.87S} \right] + \log u$$

If the discharge, $Q$, is held constant, the bracketed parts of equations 9a and 10a are constant for a given pumping test, and $W(u)$ is related to
Figure 23.—Logarithmic graph of the well function \( W(u) \)—constant discharge

\[ s = \frac{114.6Q}{T} \]  
\[ W(u) = \int_{187r^2S}^{\infty} \frac{s}{u} du \]  
\[ u = \frac{1.87r^2S}{T_f} \]
$u$ in the manner that $s$ is related to $r^2/t$. This is shown graphically in figure 24. Therefore, if values of the drawdown $s$ are plotted against $r^2/t$, or $1/t$ if only one observation well is used, on logarithmic tracing paper to the same scale as the type curve, the curve of observed data will be similar to the type curve. The data curve may then be superposed on the type curve, the coordinate axes of the two curves being held parallel, and translated to a position which represents the best fit of the field data to the type curve. An arbitrary point is selected anywhere on the overlapping portion of the sheets and the coordinates of this common point on both sheets are recorded. It is often convenient to select a point whose coordinates are both 1. These data are then used with equations 9 and 10 to solve for $T$ and $S$.

A type curve on logarithmic coordinate paper of $W(u)$ versus $1/u$, the reciprocal of the argument, could have been plotted. Values of the drawdown (or recovery), $s$, would then have been plotted versus $t$, or $t/r^2$ and superposed on the type curve in the manner outlined above. This method eliminates the necessity for computing $1/t$ values for the values of $s$.

**MODIFIED NONEQUILIBRIUM FORMULA**

It was recognized by Jacob (1950) that in the series of equation 7 the sum of the terms beyond $\log u$ is not significant when $u$ becomes small. The value of $u$ decreases as the time, $t$, increases and as $r$ decreases. Therefore, for large values of $t$ and reasonably small
values of \( r \), the terms beyond \( \log u \) in equation 7 may be neglected. When \( r \) is large, \( t \) must be very large before the terms beyond \( \log u \) in equation 7 can be neglected. Thus the Theis equation in its abbreviated or modified nondimensional form is written as

\[
s = \frac{Q}{4\pi T} \left( \log \frac{4Tt}{r^2S} - 0.5772 \right) = \frac{Q}{4\pi T} \log \frac{2.25Tt}{r^2S},
\]

which is obviously identical with equation 3. In the usual Survey units, then, this equation will be identical with equation 4, all terms being as previously defined.

In applying equation 4 to measurements of the drawdown or recovery of water level in a particular observation well, the distance \( r \) will be constant, and it follows that

at time \( t_1 \), \( s_1 = \frac{264Q}{T} \left( \log_{10} \frac{0.3Tt_1}{r^2S} \right) \);

at time \( t_2 \), \( s_2 = \frac{264Q}{T} \left( \log_{10} \frac{0.3Tt_2}{r^2S} \right) \);

and the change in drawdown or recovery from time \( t_1 \) to \( t_2 \) is

\[
s_2 - s_1 = \frac{264Q}{T} \left( \log_{10} \frac{t_2}{t_1} \right),
\]

Rewriting this equation in form suitable for direct solution of \( T \), there follows

\[
T = \frac{264Q \log_{10} \frac{t_2}{t_1}}{s_2 - s_1}, \quad (11)
\]

where \( Q \) and \( T \) are as previously defined, \( t_1 \) and \( t_2 \) are two selected times, in any convenient units, since pumping started or stopped, and \( s_1 \) and \( s_2 \) are the respective drawdowns or recoveries at the noted times, in feet.

The most convenient procedure for application of equation 11 is to plot the observed data for each well on the semilogarithmic coordinate paper, plotting values of \( t \) on the logarithmic scale and values of \( s \) on the arithmetic scale. After the value of \( u \) becomes small (generally less than 0.01) and the value of time, \( t \), becomes great, the observed data should fall on a straight line. From this straight line make an arbitrary choice of \( t_1 \) and \( t_2 \) and record the corresponding values of \( s_1 \) and \( s_2 \). Equation 11 can then be solved for \( T \). For
convenience, \( t_1 \) and \( t_2 \) are usually chosen one log cycle apart, because then

\[
\log_{10} \frac{t_2}{t_1} = 1
\]

and equation 11 reduces to

\[
T = \frac{264Q}{\Delta s}, \quad (12)
\]

where \( \Delta s \) is the change, in feet, in the drawdown or recovery over one log cycle of time.

The coefficient of storage also can be determined from the same semilog plot of the observed data. When \( s = 0 \), equation 3 becomes

\[
s = 0 = \frac{Q}{4\pi T} \log_e \frac{2.25Tt}{r^2S}.
\]

Solving for the coefficient of storage, \( S \), the equation in its final form becomes

\[
S = \frac{2.25Tt}{r^2}, \quad (13)
\]

or, in the usual Survey units,

\[
S = \frac{0.3Tt_0}{r^2}, \quad (14)
\]

where \( S \), \( T \), and \( r \) are as previously defined and \( t_0 \) is the time intercept, in days, where the plotted straight line intersects the zero-drawdown axis. If any other units were used for the time, \( t \), on the semilog plot, then obviously \( t_0 \) must be converted to days before using equation 14.

Lohman (1957) has described a simple method for determining \( S \) using the data region of the straight-line plot without extrapolating to the zero-drawdown axis.

THEIS RECOVERY FORMULA

A useful corollary to the nonequilibrium formula was devised by Theis (1935) for the analysis of the recovery of a pumped well. If a well is pumped, or allowed to flow, for a known period of time and then shut down and allowed to recover, the residual drawdown at any instant will be the same as if the discharge of the well had been continued but a recharge well with the same flow had been introduced at the same point at the instant the discharge stopped. The residual drawdown at any time during the recovery period is the difference between the observed water level and the nonpumping water level.
extrapolated from the observed trend prior to the pumping period. The residual drawdown, $s'$, at any instant will then be

$$s' = \frac{114.6Q}{T} \left[ \int_{1.87r^2S/T}^{\infty} \frac{e^{-u}}{u} du - \int_{1.87r^2S/T'}^{\infty} \frac{e^{-u}}{u} du \right]$$  \hspace{1cm} (15)$$

where $Q$, $T$, $S$, and $r$ are as previously defined, $t$ is the time since pumping started, and $t'$ is the time since pumping stopped. The quantity $1.87r^2S/Tt'$ will be small when $t'$ ceases to be small because $r$ is very small and therefore the value of the integral will be given closely by the first two terms of the infinite series of equation 7. Equation 15 can therefore be written, in modified form, in the usual Survey units, as

$$T = \frac{264Q}{s'} \log_{10} \frac{t}{t'}$$  \hspace{1cm} (16)$$

The above formula is similar in form to, and is based on the same assumptions as, the modified nonequilibrium formula developed by Jacob, and it permits the computation of the coefficient of transmissibility of an aquifer from the observation of the rate of recovery of water level in a pumped well, or in a nearby observation well where $r$ is sufficiently small to meet the above assumptions.

The Theis recovery formula is applied in much the same manner as the modified nonequilibrium formula. The most convenient procedure is to plot the residual drawdown, $s'$, against $t/t'$ on semilogarithmic coordinate paper, $s'$ being plotted on the arithmetic scale and $t/t'$ on the logarithmic scale. After the value of $t'$ becomes sufficiently large, the observed data should fall on a straight line. The slope of this line gives the value of the quantity $\log_{10} (t/t')/s'$ in equation 16. For convenience, the value of $t/t'$ is usually chosen over one log cycle because its logarithm is then unity and equation 16 then reduces to

$$T = \frac{264}{\Delta s'} Q$$  \hspace{1cm} (17)$$

where $\Delta s'$ is the change in residual drawdown, in feet, per log cycle of time. It is not possible to determine the coefficient of storage from the observation of the rate of recovery of a pumped well unless the effective radius, $r_e$, which is usually difficult to determine, is known. The Theis recovery formula should be used with caution in areas where it is suspected that boundary conditions exist. If a geologic boundary has been intercepted by the cone of depression during pumping, it may be reflected in the rate of recovery of the pumped well, and the value of $T$ determined by using the Theis recovery formula could be in error. With reasonable care the recovery in an observation well
can be used, of course, to determine both transmissibility and storage, whether or not boundaries are present.

APPLICABILITY OF METHODS TO ARTESIAN AND WATER-TABLE AQUIFERS

The methods previously discussed have been used successfully for many years in determining aquifer constants and in predicting the performance of both water-table and artesian aquifers. The derivations of the equations are based, in part, on the assumptions that the coefficient of transmissibility is constant at all times and places and that water is released from storage instantaneously with decline in head. It should be recognized, however, that these and many other idealizations are necessary before mathematical models can be used to analyze the physical phenomena associated with ground-water movement. Thus the hydrologist cannot blindly select a model, turn a crank, and accept the answers. He must devote considerable time and thought to judging how closely his real aquifer resembles the ideal. If enough data are available he will always find that no ideal aquifer, of the type postulated in the theory, could reproduce the data obtained in an actual pumping test. He should understand that the dispersion of the data is a measure of how far his aquifer departs from the ideal. Therefore, he must plan his test procedures so that they will conform as closely as possible to the theory and thus give results that can safely be applied to his aquifer. He must be prepared to find out, however, that his aquifer is too complex to permit a clear evaluation of its coefficients of transmissibility and storage. He must not tell himself or the reader that “the coefficient of storage changed” during the test but must realize that he got different values when he tried to apply his data, inconsistently, to an ideal theoretical aquifer.

Thus there is little justification for the premise that the storage coefficient of a water-table aquifer varies with the time of pumping, inasmuch as such anomalous data are merely the results of trying to apply a two-dimensional flow formula to a three-dimensional problem. The nonequilibrium formula was derived on the basis of strictly radial flow in an infinite aquifer and its application to situations where vertical-flow components occur is not justified except under certain limiting conditions. As the time of pumping becomes large, however, the rate of water-level decline decreases rapidly so that eventually the effect of vertical-flow components in water-table aquifers are minimized.

If the drawdowns are large compared to the initial depth of flow, it is necessary to adjust the observed drawdown in a pumping test of a water-table aquifer before the nonequilibrium formula is applied. According to Jacob (1944, p. 4) if the observed drawdowns are adjusted (reduced) by the factor $s^2/2m$, where $s$ is the observed draw-
down and \( m \) is the initial depth of flow, the value of \( T \) will correspond to equivalent confined flow of uniform depth, and the value of \( S \) will more closely approximate the true value. He adds that when the drawdowns are adjusted the nonequilibrium formula can be used with fair assurance even when the dewatering is as much as 25 percent of the initial depth of flow.

Where the discharging well only partially penetrates the aquifer it may also be necessary to adjust the observed drawdowns. Procedures for accomplishing this have been described by Jacob (1945).

**INSTANTANEOUS DISCHARGE OR RECHARGE**

**"BAILER" METHOD**

Skibitzke (1958) has developed a method for determining the coefficient of transmissibility from the recovery of the water level in a well that has been bailed. At any given point on the recovery curve the following equation applies:

\[
s' = \frac{V}{4\pi T t \left[ \frac{s'}{2S} \right]}
\]

where

- \( s' \) = residual drawdown,
- \( V \) = volume of water removed in one bailer cycle,
- \( T \) = coefficient of transmissibility,
- \( S \) = coefficient of storage,
- \( t \) = length of time since the bailer was removed,
- \( r_w \) = effective radius of the well.

The effective radius, \( r_w \), of the well is very small in comparison to the extent of the aquifer. As \( r_w \) is small, the term in brackets in equation 18 approaches unity as \( t \) increases. Therefore for large values of \( t \), equation 18 may be modified and rewritten, in consistent units, as

\[
s' = \frac{V}{4\pi T t} = \frac{V}{12.57 T t'}
\]

where \( s' \), \( T \), and \( t \) have units and significance as previously defined, and where \( V \) represents the volume of water, in gallons, removed during one bailer cycle. If the residual drawdown is observed at some time after completion of \( n \) bailer cycles then the following expression applies:

\[
s' = \frac{1}{12.57 T} \left[ \frac{V_1}{t_1} + \frac{V_2}{t_2} + \frac{V_3}{t_3} + \cdots + \frac{V_n}{t_n} \right],
\]

where the subscripts merely identify each cycle of events in sequence.
Thus \( V_3 \) represents the volume of water removed during the third bailer cycle and \( t_0 \) is the elapsed time from the instant that water was removed from storage to the instant at which the observation of residual drawdown was made.

If approximately the same volume of water is removed by the bailer during each cycle, then equation 20 becomes

\[
s' = \frac{V}{12.57T} \left[ \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \ldots + \frac{1}{t_n} \right].
\] (21)

The "bailer" method is thus applied to a single observation of the residual drawdown after the time since bailing stopped becomes large. The transmissibility is computed by substituting in equation 21 the observed residual drawdown, the volume of water \( V \) considered to be the average amount removed by the bailer in each cycle, and the summation of the reciprocal of the elapsed time, in days, between the time each bailer of water was removed from the well and the time of observation of residual drawdown.

"SLUG" METHOD

Ferris and Knowles (1954) discuss a convenient method for estimating the coefficient of transmissibility, under certain conditions. This is done by injecting a given quantity or "slug" of water into a well. Their equation for determining the coefficient of transmissibility is the same as the equation derived by Skibitzke for the bailer method, inasmuch as the effects of injecting a slug of water into a well are identical, except for sign, with the effects of bailing out a slug of water. Thus equation 19 has direct application, only \( s' \) now represents residual head, in feet, at the time \( t \), in days, following injection of \( V \) gallons of water.

As used in the field, this method requires the sudden injection of a known volume of water into a well and the collection thereafter of a rapid series of water-level observations to define the decay of the head that was built up in the well. An arithmetic plot of residual head values versus the reciprocals of the times of observation should produce a straight line whose slope, appropriately substituted in equation 19, permits computation of the transmissibility.

Suggested equipment for use in injecting a slug of water into a well, and for making the rapid series of water-level observations required immediately thereafter, is shown schematically in figure 25.

The duration of a "slug" test is very short, hence the estimated transmissibility determined from the test will be representative only of the water-bearing material close to the well. Serious errors will
be introduced unless the observation well is fully developed and completely penetrates the aquifer. Use of the "slug" test should probably be restricted to artesian aquifers of small to moderate transmissibility (less than 50,000 gallons per day per foot).
Controlled pumping tests have proved to be an effective tool in determining the coefficients of storage and transmissibility. In the usual test the discharge rate of the pumped well is held constant, whereas the drawdown varies with time. The resulting data are analyzed graphically as previously described. Jacob and Lohman (1952) derived a formula for determining the coefficients of storage and transmissibility from a test in which the discharge varies with time and the drawdown is held constant. The formula, based on the assumptions that the aquifer is of infinite areal extent, and that the coefficients of transmissibility and storage are constant at all times and all places, is developed from the analogy between the hydrologic conditions in an aquifer and the thermal conditions in an equivalent thermal system. The formula is written as

\[ Q = 2 \pi T \delta_w G(\alpha), \]  

where

\[ G(\alpha) = \frac{4 \alpha}{\pi} \int_0^\infty xe^{-\alpha x} \left[ \frac{\pi}{2} + \tan^{-1} \frac{Y_0(x)}{J_0(x)} \right] dx \]

and

\[ \alpha = \frac{Tt}{r_w^2 s_w}. \]

Using the customary Survey units, equations 22 and 24 are rewritten in the form

\[ Q = \frac{T \delta_w G(\alpha)}{229} \]

and

\[ \alpha = \frac{0.134 Tt}{r_w^2 s_w} \]

where \( Q, T, \) and \( t \) have the units and meaning previously defined and where

\( \delta_w = \) constant drawdown, in feet, in the discharging well,

\( r_w = \) effective radius, in feet, of the discharging well.

The terms \( J_0(x) \) and \( Y_0(x) \) are Bessel functions of zero order of the first and second kinds respectively.

The integration required in equation 23 cannot be accomplished directly so it is necessary to replace the integral with a summation and solve it by numerical methods. In this fashion values of \( G(\alpha) \) for values of \( \alpha \) from \( 10^{-4} \) to \( 10^{12} \), have been tabulated by Jacob and Lohman, (1952), and are given herewith in table 3. The term \( G(\alpha) \) is
here designated as the "well function of α, constant-head situation." This table is used in the same manner as table 2, which gives values of \( W(u) \) versus \( u \).

It is seen from equations 25 and 26 that if \( Q \) can be measured for several values of \( t \) and if the constant drawdown, \( s_{w} \), and the effective radius, \( r_{w} \), are known, \( S \) and \( T \) can be determined. It is not possible to determine \( S \) and \( T \) directly, however, since \( T \) occurs both in the argument of the function and as a multiplier of \( G(\alpha) \). A convenient graphical method, similar to that used in solving the nonequilibrium formula, makes it possible to obtain a simple solution.

The first step in this method is the plotting of a type curve on logarithmic coordinate paper. From table 3, values of \( G(\alpha) \) were plotted against the argument \( \alpha \) to form the type curve shown in figure 26. It is shown in several segments in order that the entire type curve may be plotted on a sheet of convenient size. Rearranging equations 25 and 26 there follows:

\[
Q = \frac{T s_{w}}{229} G(\alpha)
\]

or

\[
\log Q = \left[ \log \frac{T s_{w}}{229} \right] + \log G(\alpha), \tag{27}
\]

and

\[
t = \frac{r_{w}^{2} S}{0.13 T^{2}}
\]

or

\[
\log t = \left[ \log \frac{r_{w}^{2} S}{0.13 T^{2}} \right] + \log \alpha. \tag{28}
\]

If the drawdown, \( s_{w} \), is held constant, the bracketed parts of equations 27 and 28 are constant for any given test and \( \log G(\alpha) \) is related to \( \log \alpha \) in the same manner that \( \log Q \) is related to \( \log t \). (Note the similarity in form between equations 27 and 28 and equations 9a and 10a.) Therefore if values of the discharge, \( Q \), are plotted against corresponding values of time, \( t \), on logarithmic tracing paper to the same scale as the type curve, the curve of observed data will be similar to the type curve. The data curve may then be superposed on the type curve, the coordinated axes of the two curves being held parallel, and translated to a position that represents the best fit of the data to the type curve. An arbitrary point is selected on the overlapping portion of the sheets and the coordinates of this common point on both sheets are used with equations 25 and 26 to solve for \( T \) and \( S \). This graphical solution is similar to that used with the Theis nonequilibrium formula.
Figure 26.—Logarithmic graph of well function $G(\alpha)$—constant drawdown.
Jacob and Lohman (1952) showed that for large values of \( t \), the function \( G(\alpha) \) can be replaced by \( 1/W(u) \), and it has already been shown (see discussion, p. 99) that the approximate form of \( W(u) \) is given by \( 2.30 \log_{10} (2.25Tt/Sr^2) \). Making this substitution for \( G(\alpha) \) in equation 22, there follows

\[
Q = \frac{4\pi Ts_w/2.30}{\log_{10} (2.25Tt/r_w^2S)}
\]

or, rearranging terms,

\[
\frac{s_w}{Q} = 2.30 \frac{\log t}{4\pi T} + \frac{2.30}{4\pi T} \log \frac{2.25T}{S}.
\]

It should be evident from the form of equation 29, that if arithmetic values of the variable \( s_w/Q \) are plotted against logarithmic values of the variable \( t/r_w^2 \) the points will define a straight line. The slope of this line, in equation 29, is the prefix of the variable term \( \log (t/r_w^2) \). In other words,

\[
\text{Slope of straight-line plot} = \frac{\Delta(s_w/Q)}{\Delta \log (t/r_w^2)} = \frac{2.30}{4\pi T}.
\]

Once the slope of the graph is determined, therefore, the coefficient of transmissibility may be computed from the relation

\[
T = \frac{2.30 \Delta(\log t/r_w^2)}{4\pi \Delta(s_w/Q)}.
\]
If the slope is measured over one log cycle then the term $\Delta \log \left(\frac{t}{r_2^2}\right)$ equals unity and equation 30 is further simplified to the form

$$T = \frac{2.30}{4\pi \Delta(s_w/Q)}.$$  \hspace{1cm} (31)

The coefficient of storage could then be found by substituting in equation 29 the computed value of $T$ and the coordinates of any convenient point on the straight-line plot. However, the computation is greatly simplified by noting that for the point where the straight-line plot intersects the logarithmic time axis (that is, where $s_w/Q = 0$), equation 29 becomes

$$S = 2.25 T(t/r_2^2)_0.$$  \hspace{1cm} (32)

In the usual Survey units, equations 31 and 32 are written

$$T = \frac{264}{\Delta(s_w/Q)}$$  \hspace{1cm} (33)

and

$$S = 0.3 T(t/r_2^2)_0.$$  \hspace{1cm} (34)

Thus equations 33 and 34 are applied through the simple device of a semilogarithmic plot where values of $s_w/Q$ are plotted on the arithmetic scale against corresponding values of $t/r_2^2$ on the logarithmic scale.

The methods that have been outlined in this section are useful in determining the coefficient of transmissibility but should be used with caution in determining the coefficient of storage because it is often difficult to determine the effective radius of the pumped well.

**CONSTANT DISCHARGE WITH VERTICAL LEAKAGE**

"LEAKY AQUIFER" FORMULA

A problem of practical interest is that of an elastic artesian aquifer that is replenished by vertical leakage through overlying or underlying semipermeable confining beds. In most places the confining beds only impede or retard the movement of ground water rather than prevent it. It is often true that this retardation of ground-water movement is sufficient so that the Theis equation (which assumes impermeable confining beds) can be applied. Nevertheless there will be occasions when departure of the test data from the predictions of the Theis equation will require investigation of the ability of the confining beds to transmit water.

As an example of the magnitude of flow through material of low permeability, consider a semipermeable confining bed, 50 feet thick, consisting of silty clay that has a permeability of 0.2 gallon per day per square foot. Such a material is listed by Wenzel (1942, p. 13,
lab. no. 2,278) as including about 49 percent (by weight) clay and about 45 percent silt. Assume that the confining bed is saturated and that in some manner there is established and maintained a head differential of 25 feet between the top and bottom surfaces of the bed. The rate of percolation, related to this head differential, through the confining bed is computed from the previously given (see p. 73) variant of Darcy's law,

\[ Q_d = P'IA, \]

where, in this example,

- \( Q_d \) = discharge in gallons per day through specified area of confining bed,
- \( P' \) = vertical permeability of confining bed = 0.2 gallon per day per square foot,
- \( I \) = hydraulic gradient imposed on confining bed = 25/50 = 0.5 foot per foot,
- \( A \) = specified area of confining bed through which percolation occurs.

Thus, through a confining-bed area of one square foot,

\[ Q_d = 0.2 \times 0.5 \times 1 = 0.1 \text{ gallon per day}, \]

or, through a confining-bed area of one square mile,

\[ Q_d = 0.2 \times 0.5 \times 5,280 \times 5,280 = 2,800,000 \text{ gallons per day}. \]

It is known that the cone of depression created by pumping a well in an artesian aquifer grows rapidly and thus in a relatively short time encompasses a large area. As shown by the above computations, the total amount of vertical seepage through confining beds may be quite large, even though the permeability of these formations is relatively small. If the confining bed in turn is overlain by an aquifer of appreciable storage and transmitting capacity, the radius of the cone of influence developed by a well pumping from the artesian aquifer will be determined by the hydrologic regimen of the artesian aquifer, the confining bed, and the leakage-source aquifer.

The first detailed analysis and solution of the leaky-aquifer problem was developed by DeGlee (1930) and later supplemented by Steggwentz and Van Nes (1939).

In these analyses, assumptions related to the physical flow system are: (a) the artesian aquifer is bounded above or below by a semipermeable confining bed, (b) the aquifer, when pumped is supplied by leakage through the confining bed, the leakage being proportional to the drawdown, and (c) the aquifer and confining bed are independently homogeneous and isotropic. It is also assumed that the water level in the aquifer supplying water to the semipermeable bed is maintained

---

at or very near static level through the interval of pumping. The solution developed is for the steady-state condition, wherein it is assumed that the drawdown is zero at \( r = \infty \).

Jacob (1946) also analyzed this problem, verifying the solution for steady flow and also developing a solution for the transient state. His final steady-state equation, in nondimensional form, for the drawdown in an infinite artesian aquifer has the form

\[
s = \frac{Q}{2\pi T} K_0(x)
\]

or, in the usual Survey units,

\[
s = \frac{229 Q K_0(x)}{T},
\]

where

\[
x = \frac{br}{a}
\]

and

\[
a = \sqrt{\frac{T}{S}}
\]
\[
b = \sqrt{\frac{P'}{m'S}}
\]

\( T = \) coefficient of transmissibility of the artesian aquifer in gallons per day per foot,

\( P' = \) coefficient of vertical permeability of the semipermeable confining bed, in gallons per day per square foot,

\( S = \) coefficient of storage of the artesian aquifer,

\( Q = \) rate of withdrawal by the pumped well, in gallons per minute,

\( m' = \) thickness of the semipermeable confining bed, in feet,

\( r = \) distance from the pumped well to the observation well, in feet,

\( s = \) drawdown in the observation well, in feet.

The symbol \( K_0(x) \) is a notation widely but not universally used to identify the modified Bessel function of the second kind of the zero order. In order to avoid any misunderstanding of its present usage it is identified as follows:

\[
K_0(x) = -[0.5772 + \log_e (x/2)] I_0(x) + (1/1!)^2(x/2)^2 + (1/2!)^2(x/2)^4(1+1/2) + \ldots,
\]
where

\[ I_0(x) = 1 + \frac{(x/2)^2}{1!} + \frac{(x/2)^4}{2!} + \frac{(x/2)^6}{3!} + \ldots \] (39)

The notation \( I_0(x) \) is used to represent the modified Bessel function of the first kind of zero order. Values of the function \( K_0(x) \) over the range of interest for most ground-water problems are given in table 4.

Equations 36 and 37 may be rewritten in the following form:

\[ \log s = \log \left[ \frac{229 \times Q}{T} \right] + \log K_0(x) \] (40)
\[ \log r = \log \left[ \frac{a}{b} \right] + \log x \] (41)

The bracketed portions of equations 40 and 41 include all the terms that have been assumed constant in the derivation. It follows then that the variable \( s \) is related to \( r \) in the same manner that \( K_0(x) \) is related to \( x \). Thus the form of equations 40 and 41 once again suggests the same convenient method of graphical solution that has already been described for resolving the Theis formula. A type curve for use in solving equations 36 and 37 is prepared by plotting on logarithmic graph paper the values given in table 4. In figure 27 curve AA is in part a duplication of the lower part of curve BB and in part an extension of that curve into the next lower log cycle.

The solution of equations 36 and 37 thus requires plotting the field observations of \( s \) and \( r \), at some particular time \( t \), on logarithmic graph paper, using the same size of logarithmic scale adopted for the type curve. The data curve is superposed on the type curve, the coordinate axes of the two curves being held parallel, and translated to the position that represents the best fit of the field data to the type curve. When the match position is found, the amount of shift or translation from the \( s \) scale to the \( K_0(x) \) scale is measured by the bracketed term of equation 40, and the translation between the \( r \) scale and the \( x \) scale is represented by the bracketed member of equation 41. An arbitrary point is selected on the data curve and the coordinates of this common point on both the data curve and the type curve are recorded. These coordinates, when substituted in equations 36 and 37, permit computation of the coefficient of transmissibility, \( T \), of the artesian bed, and the value of \( x \), which has inherent in it the coefficient of vertical permeability of the leaky confining bed.
\[ x = \frac{229 \cdot Q \cdot K_0(x)}{Q} \]

\[ \frac{b}{h} = \sqrt{\frac{Q}{Q_m}} \]

**Figure 27.** Logarithmic graph of the modified Bessel function \( K_0(x) \).
### Table 4.—Values of \(K_0(x)\), the modified Bessel function of the second kind of zero order, for values of \(x\) between 10^{-1} and 9.9

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1 When \(z=0\), \(K_0(z)\) = \(ae\).

In application it is not possible to determine either \(a\) or \(b\) from field observation of steady flow, but their ratio can be determined from the definition of \(x\):

\[
x = r(b/a) = r\sqrt[4]{P'/m'S/T'/S} = r\sqrt[4]{P'/m'^*}.
\] (42)

The vertical permeability of the leaky bed can thus be determined from equation 42 if the bed thickness, \(m'\), is known. However, \(S\), the coefficient of storage for the artesian aquifer cannot be determined as it is removed from the \(b/a\) ratio by cancellation. Hantush (1955) has designated the ratio \(P'/m'\) as the "leakage coefficient," and
Hantush and Jacob (1955) have described in considerable detail their development of equations for the nonsteady-state solution to the foregoing problem.

The preceding discussion has stipulated that equations 36, 37, and 42 are properly applied only to steady-state conditions. This means that enough time must have elapsed for the drawdown to have stabilized throughout the region for which the plot of $s$ versus $r$ is to be made. The manner in which the drawdown stabilizes at observation points at selected distances from the discharging well is shown on a semilogarithmic plot by Hantush and Jacob (1955, fig. 1). In effect their plot shows individual time-drawdown curves because values of drawdown divided by a constant are plotted against values of the logarithm of time multiplied by a constant. Of special interest is the fact that for all the curves, regardless of the represented distance from the discharging well, the drawdown stabilizes or levels off at the same value of time.

Assuming, therefore, that the requirement of stabilized drawdown has been met, an important feature of the logarithmic type curve (fig. 27) should be recognized. Note that the curve is drawn only for values of $x$ greater than 0.01. Thus the matching of a logarithmic plot of $s$ versus $r$ against the leaky-aquifer type curve is appropriate only if the observed data and computed results can be shown to yield values of $x$ (which is directly related to $r$) that are greater than 0.01. Actually the critical value of $x$ is about 0.03, as can be demonstrated in the following manner.

In the tables of the Bessel functions (U.S. Department of Commerce, 1952) the following relation applies for small values of $x$:

$$K_0(x) = E_0(x) + F_0(x) \log_{10}(x).$$

The tables show that for values of $x$ ranging from 0 to about 0.03 the values of $E_0(x)$ and $F_0(x)$ are 0.116 and -2.303 respectively. Substituting these equivalents in the above relation yields

$$K_0(x) = 0.116 - 2.303 \log_{10}(x),$$

which, by substitution from equation 37 and conversion to the natural logarithm, becomes

$$K_0(x) = 0.116 - \log_e (br/a).$$

If equation 35 is rewritten in terms of the difference in drawdown between two points at radii $r_1$ and $r_2$ (where $r_2 > r_1$) on the cone of
depression, and the foregoing relation for $K_o(x)$ substituted therein, there follows the expression

$$s_1 - s_2 = \frac{Q}{2\pi T} \left[ (0.116 - \log_e \frac{br_1}{a}) - (0.116 - \log_e \frac{br_2}{a}) \right],$$

or

$$s_1 - s_2 = \frac{Q}{2\pi T} \log_e \frac{r_2}{r_1},$$

which is the familiar Thiem equilibrium formula previously presented in the form of equation 1. The conclusion to be drawn is that in the region $x < 0.03$ a logarithmic plot of $s$ versus $r$ exhibits only the effects of radial flow through the aquifer toward the discharging well; the leakage effects are not significant enough to influence the shape of the curve. Although the leaky-aquifer type curve could be extended readily into this region of low $x$ values, its curvature is insensitive to leakage and is too slight to permit a matching that would be definitive of the $x$ value needed for computing the leakage coefficient.

The nature of the abbreviated relation for $K_o(x)$, presented in the preceding discussion, suggests a simple means for analyzing the steady-state drawdown data within the region $x < 0.03$. Rewriting equation 35 in terms of this special relation for $K_o(x)$ produces

$$s = \frac{Q}{2\pi T} [0.116 - \log_e (br/a)],$$

or, in the usual Survey units and the common logarithm,

$$s = \frac{229Q}{T} \left( 0.116 - 2.303 \log_{10} \frac{br}{a} \right). \quad (43)$$

Recognizing that $s$ and $r$ are the only variables in equation 43, obviously a semilogarithmic plot of $s$ versus $\log r$ produces a straight line. If $r_0$ is the intercept of this straight line at the zero-drawdown axis, appropriate substitution in equation 43 yields

$$\log_{10} \frac{br_0}{a} = \frac{0.116}{2.303},$$

or

$$\frac{b}{a} = \frac{1.12}{r_0}. \quad (43a)$$
The analysis of steady-state test data for a leaky aquifer can thus be summarized in the following three simple procedures:
1. Select for plotting only the drawdown data which are within the region where drawdowns have levelled off.
2. Use equations 36, 37, and 42 with a logarithmic plot of $s$ versus $r$, matched to the leaky-aquifer type curve (fig. 27), only if the observed data and resulting computations produce values of $x$ greater than 0.03.
3. Use equations 2, 4, and 43a with a semilogarithmic plot of $s$ versus $\log r$ if the data and resulting computations produce values of $x$ less than 0.03.

The earliest observations of drawdown in each observation well, when $s$ is small, should conform to the Theis nonequilibrium type curve for the infinite (nonleaky) aquifer if the rate of leakage from the confining bed is comparatively small. The coefficient of storage for the artesian aquifer can be determined under these conditions from the earliest observations of drawdown (Jacob, 1946, p. 204). The computed coefficient of transmissibility should be checked by comparing the value obtained from matching the earliest data to the nonequilibrium type curve with the value obtained by matching the later data to the steady-state leaky-aquifer type curve. If consistency of the $T$ values is not obtained, then the leakage may be causing too much deviation at the smaller values of $t$ to permit application of the Theis nonequilibrium formula.

**VARIABLE DISCHARGE WITHOUT VERTICAL LEAKAGE**

By R. W. Stallman

CONTINUOUSLY VARYING DISCHARGE

The rate at which water is pumped from a well or well field commonly varies with time in response to seasonal changes in demand. For instance, the pumping rate, as shown by records of daily or monthly discharge, is often found to be varying continuously. Where this element of variability is recognized in ground-water problems, the analytical methods that are described in the preceding sections of this report are not applicable without some modification or approximation. Exact equations could perhaps be developed for the case of continuously varying discharge, but the cost of analysis, in terms of time and effort, would likely be prohibitive considering that a separate and specific solution would be required for each problem. It is considered more expedient, therefore, to utilize the existing analytical methods, rendering them applicable to the field situation by introducing tolerable approximations of the field conditions. As an example, consider a situation where the pumping rate in a well (which may also represent a well field) tapping an artesian aquifer varies continuously with time in the manner indicated by the smooth curve shown in